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Symmetry reductions of a generalized, cylindrical nonlinear Schrödinger equation

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Abstract. In this paper, symmetry reductions for a generalized, cylindrical nonlinear Schrödinger equation are presented. These are obtained using an extension of the *direct method*, originally developed by Clarkson and Kruskal, which involves no group theoretic techniques.

1. Introduction

In this paper we discuss symmetry reductions of a generalized, cylindrical nonlinear Schrödinger (GCNLS) equation

$$\begin{aligned} \mathrm{i}u_t + u_{\rho\rho} + \rho^{-1}u_\rho - \kappa^2\rho^{-2}u + (a_1 + \mathrm{i}a_2)(|u|^2u)_\rho \\ + (b_1 + \mathrm{i}b_2)u(|u|^2)_\rho + cu|u|^4 + du|u|^2 \\ = 0 \end{aligned} \tag{1.1}$$

with κ, a_1, a_2, b_1, b_2, c and d constants, not all zero. This equation is the special case of the $(2+1)$-dimensional generalized nonlinear Schrödinger equation

$$\begin{aligned} \mathrm{i}\psi_t + \psi_{xx} + \psi_{yy} + (\alpha_1 + \mathrm{i}\alpha_2)\cdot\nabla(\psi|\psi|^2) \\ + (\beta_1 + \mathrm{i}\beta_2)\cdot\psi\nabla(|\psi|^2) + c\psi|\psi|^4 + d\psi|\psi|^2 \\ = 0 \end{aligned} \tag{1.2}$$

with α_1, α_2, β_1, β_2 constant vectors, c, d constants and where $\nabla\phi = (\phi_x, \phi_y)$, which arises in quantum field theory (Clarkson and Tuszyński 1990, Dixon and Tuszyński 1989, Tuszyński and Dixon 1989). Setting $\psi(x,y,t) = u(\rho,t)\exp(\mathrm{i}\kappa\theta)$, with $\rho = (x^2+y^2)^{1/2}$ and $\theta = \tan^{-1}(y/x)$, in (1.2) yields the GCNLS equation (1.1).

The GCNLS equation (1.1) occurs, in some form, in a wide variety of physical applications: in wave propagation in nonlinear and dispersive media, in quantum field theory, the time-dependent Landau–Ginzberg model of phase transitions and propagations of slowly varying electromagnetic wave envelopes in plasma, in the theory of weakly nonlinear dispersive water waves, and in the nonlinear dynamics of superfluid films for which u is the condensate wavefunction, related to the film thickness and to the superfluid velocity. Of particular interest is the so-called cylindrical nonlinear Schrödinger (CNLS) equation

$$iu_t + u_{\rho\rho} + \rho^{-1}u_\rho \pm |u|^2u = 0 \tag{1.3}$$

which plays an important role in the theory of light wave envelopes in dispersive media with nonlinear refractive index (Hasegawa 1990). Furthermore, the CNLS equation (1.3) is the so-called 'equation of self-focusing'. It is known that there exist solutions of (1.3) which have a singularity in finite time and there has been much interest in the determination of the structure of this singularity (cf Ablowitz and Segur 1979, Landman*et al* 1988, 1991, LeMesurier*et al* 1988a, b, Malkin and Shapiro 1991, Rasmussen and Rypdal 1986, Rypdal *et al* 1985, Rypdal and Rasmussen 1986, Smirnov and Fraiman 1991, Wood 1984, Zakharov and Synath 1976).

Specifically, optical solitons (or wave packets), a manifestation of this nonlinearity, find application in distortionless signal transmission along optical fibres. Such applications occur in long-distance data transmission (~ 10 000 km), removing the need for expensive en-route 'repeater' equipment, e.g., in telecommunications, and more recently in the field of high-speed ultra-low-noise data transmission. This latter application makes use of a property known as 'squeezing': data are encoded on one of a pair of conjugate quantum observables in which noise is naturally reduced, at the expense of increased, but inconsequential, noise in the other variable (Abram and Padjen 1991).

The classical method for finding symmetry reductions of PDES is the Lie group method of infinitesimal transformations (cf Bluman and Cole 1974, Bluman and Kumei 1989, Hill 1982, Olver 1986), for which symbolic manipulation programmes have been developed (a survey of the different packages presently available is given by Champagne *et al* 1991).

There have been several generalizations of the classical Lie method for symmetry reductions, in particular the non-classical method of group-invariant solutions, in the folllowing referred to as the *non-classical method*, due to Bluman and Cole (1969).

Clarkson and Kruskal (1989), hereafter referred to as CK, developed a direct and algorithmic method for finding symmetry reductions (in the following referred to as the 'direct method') which is used to obtain previously unknown reductions of the Boussinesq equation. Levi and Winternitz (1989) subsequently gave a group theoretical explanation of these results by showing that all the new reductions of the Boussinesq equation could also be obtained using the non-classical method of Bluman and Cole (1969). The novel characteristic about the direct method is that it involves no use of group theory. It has been employed to obtain new symmetry reductions and exact solutions for several physically significant PDES (Clarkson 1989a, b, 1990, 1992, Clarkson and Hood 1992, Clarkson and Winternitz 1991, Lou 1990a, b, 1991, Lou and Ni 1991, Lou *et al* 1991, Nucci and Clarkson 1992), which represent significant progress.

There is much current interest in the determination of symmetry reductions of PDES which reduce the equations to ODES. One then checks if the resulting ODE is of *Painlevé type* (i.e. whether its solutions have no movable singularities other than poles). It appears to be the case that whenever the ODE is of Painlevé type then it can be solved explicitly, leading to exact solutions to the original equation. Conversely, if the resulting ODE is not of Painlevé type, then often one is unable to solve it explicitly.

In this paper we apply the direct method to the GCNLS equation (1.1) and obtain some new reductions for this equation and consequently also for (1.2).

2. Physical and mathematical background

In the first part of this section we give a brief interpretation of the GCNLS equation (1.1), in the context of 'solitons'† in optical fibres; the second part consists of a short discussion of mathematical properties of the equation.

2.1. *Physical background*

We first consider the CNLS equation (1.3), which is the model equation describing envelope soliton propagation. Here t represents the 'distance' along the direction of propagation, and ρ, the 'time' in the group velocity frame. The second and third terms originate from the dispersion of the group velocity, that is the group velocity is dependent on the wavelength, producing dispersion of the wave. The nonlinearity is produced by the dependence of the wavelength on the intensity, u of the wave. This is due to the 'Kerr effect' that is the dependence of the refractive index of a dialectric material on the (square of the modulus of the) electric field component of the electromagnetic field u. In fact, the refractive index n, is given by $n = n_0(\omega) + n_2|E|^2$, where ω is the angular frequency of the light, E is the electric field and n_2 is known as the 'Kerr coefficient' (Hasegawa 1990). In order to model further physical effects, more terms may be added to (1.3). The fourth term in (1.1), namely $(a_1 + \mathrm{i}a_2)(|u|^2u)_\rho$, models nonlinear dispersion generated by the dependence on wavelength of the Kerr coefficient.

One may also model dissipation originating from Raman scattering in the fibre: if a sample is illuminated by monochromatic light, and the scattered light observed, the observed spectrum consists of a strong line and weaker lines on either side. These lines are interpreted as a manifestation of molecular vibrational transitions, i.e. the energy of the emitted photon may be altered from that of the incident photon because of energy level transitions of molecules in the sample (Elliot and Dawber 1979, Ashcroft and Mermin 1976).

2.2. *Mathematical background*

Symmetry reductions of the $(3+1)$-dimensional generalized nonlinear Schrödinger (GNLS) equation

$$\mathrm{i}\psi_t + \Delta\psi + (\alpha_1 + \mathrm{i}\alpha_2)\cdot\nabla(\psi|\psi|^2) + (\beta_1 + \mathrm{i}\beta_2)\cdot\psi\nabla(|\psi|^2) + c\psi|\psi|^4 + d\psi|\psi|^2 = 0 \tag{2.1}$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2$ constant vectors, c, d constants and where $\Delta\phi = \phi_{xx} + \phi_{yy} + \phi_{zz}$ and $\nabla\phi = (\phi_x, \phi_y, \phi_z)$ have been obtained by Clarkson and Tuszyński (1990) and Clarkson (1992) who looked for plane wave reductions of (2.1), i.e. reductions of the form

$$\psi(x, y, z, t) = \beta(t)R(\xi)\exp\{\mathrm{i}[\Theta(\xi) + \phi(x, y, z, t)]\}$$

† Strictly, the term soliton refers only to solutions of integrable equations. For convenience, we relax the definition so as to include solutions of non-integrable equations.

with $\xi = (n_1x + n_2y + n_3z)\eta(t)$ and where $\beta(t)$, $\eta(t)$ and $\phi(x, y, z, t)$ are specified functions and $\boldsymbol{n} = (n_1, n_2, n_3)$ is a constant unit vector. In these reductions, $R(\xi)$ and $\Theta(\xi)$ were expressed in terms of the second and fourth Painlevé transcendents (cf Ince 1956), Jacobi and Weierstrass elliptic functions, Airy functions and parabolic cylinder functions, for various special choices of the parameters α_1, α_2, β_1, β_2, c, d and $\boldsymbol{n}$.

In a series of papers, Gagnon and Winternitz (1988, 1989a) and Gagnon *et al* (1989) classified symmetry reductions of the $(3+1)$-dimensional cubic and quintic nonlinear Schrödinger equations

$$\mathrm{i}\psi_t + \Delta\psi + c\psi|\psi|^4 + d\psi|\psi|^2 = 0 \tag{2.2}$$

with c and d constants, not both zero, which is the special case of (2.1) where $\alpha_1 = \alpha_2 \equiv \mathbf{0}$ and $\beta_1 = \beta_2 \equiv \mathbf{0}$, using the classical Lie method. Subsequently, Gagnon and Winternitz (1989b, c) considered symmetry reductions of the cylindrical cubic and quintic nonlinear Schrödinger equations

$$\mathrm{i}\psi_t + \psi_{\rho\rho} + \rho^{-1}\psi_\rho + \rho^{-2}\psi_{\theta\theta} + c\psi|\psi|^4 + d\psi|\psi|^2 = 0 \tag{2.3}$$

and the spherical cubic and quintic nonlinear Schrödinger equations

$$\mathrm{i}\psi_t + \psi_{\rho\rho} + 2\rho^{-1}\psi_\rho + \rho^{-2}\psi_{\theta\theta} + c\psi|\psi|^4 + d\psi|\psi|^2 = 0. \tag{2.4}$$

In these studies, the authors obtained solutions of (2.2) and (2.3) expressible in terms of the second and fourth Painlevé transcendents, elementary functions and Jacobi elliptic functions and solutions of (2.4) expressible in terms of elementary and Jacobi elliptic functions.

Tajiri (1983), Gagnon (1990) and Gagnon and Paré (1991) have discussed symmetry reductions of the $(2+1)$-dimensional cubic nonlinear Schrödinger equation

$$\mathrm{i}\psi_t + \psi_{xx} + \psi_{yy} + d\psi|\psi|^2 = 0 \tag{2.5}$$

using the classical Lie method. In particular, Gagnon (1990) and Gagnon and Paré (1991) obtained symmetry reductions which were not included in the earlier classification for (2.2) and (2.3) (see also Giannini and Joseph 1991).

Symmetry reductions of the $(1+1)$-dimensional GNLS equation

$$\mathrm{i}\psi_t + \psi_{xx} + (a_1 + \mathrm{i}a_2)(\psi|\psi|^2)_x + (b_1 + \mathrm{i}b_2)\psi(|\psi|^2)_x + c\psi|\psi|^4 + d\psi|\psi|^2 = 0 \tag{2.6}$$

with a_1, a_2, b_1, b_2, c, d constants, have been discussed by Florjańczyk and Gagnon (1990, 1992) using the classical Lie method and by Clarkson (1992) using the direct method. Solutions were obtained expressible in terms of the second and fourth Painlevé transcendents, Jacobi and Weierstrass elliptic functions, Airy functions and parabolic cylinder functions

3. Symmetry reductions obtained by the direct method

In order to obtain symmetry reductions of (1.1), we write it as the system

$$\mathrm{i}u_t+u_{\rho\rho}+\rho^{-1}u_\rho-\kappa^2\rho^{-2}u+(a_1+\mathrm{i}a_2)(u^2v)_\rho+(b_1+\mathrm{i}b_2)u(uv)_\rho+cu^3v^2+du^2v=0 \tag{3.1a}$$

$$-\mathrm{i}v_t+v_{\rho\rho}+\rho^{-1}v_\rho-\kappa^2\rho^{-2}v+(a_1-\mathrm{i}a_2)(uv^2)_\rho+(b_1-\mathrm{i}b_2)v(uv)_\rho+cu^2v^3+duv^2=0 \tag{3.1b}$$

where v is the (formal) complex conjugate of u. To apply the direct method to this system of equations, following Clarkson (1992), we make the ansatz

$$u(\rho,t)=\beta(t)P(z)\exp\{\mathrm{i}\Phi(\rho,t)\} \tag{3.2a}$$

$$v(\rho,t)=\beta(t)Q(z)\exp\{-\mathrm{i}\Phi(\rho,t)\} \tag{3.2b}$$

with $z=\rho\eta(t)+\zeta(t)$ and where $\beta(t)$, $\eta(t)$, $\zeta(t)$ and $\Phi(\rho,t)$ are (real) functions to be determined (and we shall also assume that $z_x\not\equiv 0$, i.e. $\eta\not\equiv 0$). We remark that as in the original application of the direct method in CK, it is sufficient to seek solutions in the linear form (3.2) rather than use the more general ansatz

$$u(\rho,t)=U(\rho,t,P(z)) \qquad v(\rho,t)=V(\rho,t,Q(z))$$

as can easily be shown. Substituting (3.2) into (3.1*a*) yields

$$\begin{aligned}\beta\eta^2P''&+(a_1+\mathrm{i}a_2)\beta^3\eta(2P^2Q'+PQP')+(b_1+\mathrm{i}b_2)\beta^3\eta(P^2Q'+PQP')+c\beta^5P^3Q^2\\&+[(\mathrm{i}a_1-a_2)\Phi_\rho+d]\beta^3P^2Q+\left[\frac{\beta\eta}{\rho}+\mathrm{i}\beta\left(\rho\frac{\mathrm{d}\eta}{\mathrm{d}t}+2\eta\Phi_\rho+\frac{\mathrm{d}\zeta}{\mathrm{d}t}\right)\right]P'\\&+\left[\mathrm{i}\left(\frac{\beta}{\rho}\Phi_\rho+\beta\Phi_{\rho\rho}+\frac{\mathrm{d}\beta}{\mathrm{d}t}\right)-\beta\left(\Phi_t+\Phi_\rho^2+\frac{\kappa^2}{\rho^2}\right)\right]P=0\end{aligned} \tag{3.3}$$

with $'\equiv \mathrm{d}/\mathrm{d}z$. In order that this is an ODE we require that ratios of coefficients of powers and derivatives of P and Q be some function of z (to be determined). We must also consider the equation produced by substituting our ansatz into (3.1*b*). This is equivalent to interchanging P and Q in (3.3), and more significantly, letting $\mathrm{i}\to-\mathrm{i}$; thus the real and imaginary parts of each coefficient of (3.3) have to be considered separately. There are three cases to consider: (i) $\beta(t)$ and $\eta(t)$ both constants; (ii) $\beta^2(t)=\eta(t)$, with $a_1=a_2=d=0$; (iii) $\beta(t)=\eta(t)$, with $a_1=a_2=b_1=b_2=c=0$.

3.1. *β and η both constants*

Without loss of generality, we set $\eta=1$ and $\beta=1$, then (3.3) reduces to

$$\begin{aligned}P''&+(a_1+\mathrm{i}a_2+b_1+\mathrm{i}b_2)(P^2Q'+PQP')+cP^3Q^2+[(\mathrm{i}a_1-a_2)\Phi_\rho+d]P^2Q\\&+\left[\frac{1}{\rho}+\mathrm{i}\left(2\Phi_\rho+\frac{\mathrm{d}\zeta}{\mathrm{d}t}\right)\right]P'\\&+\left[\mathrm{i}\left(\frac{1}{\rho}\Phi_\rho+\Phi_{\rho\rho}\right)-\left(\Phi_t+\Phi_\rho^2+\frac{\kappa^2}{\rho^2}\right)\right]P=0.\end{aligned} \tag{3.4}$$

From the coefficient of P' in (3.4) we have

$$\frac{1}{\rho} + 2\mathrm{i}\Phi_\rho + \mathrm{i}\frac{\mathrm{d}\zeta}{\mathrm{d}t} = \Gamma_{11}(z) + \mathrm{i}\Gamma_{12}(z) \tag{3.5}$$

where $\Gamma_{11}(z)$ and $\Gamma_{12}(z)$ are (real) arbitrary functions to be determined. Without loss of generality, we set $\Gamma_{12} = 0$ (due to a scaling freedom in P), and so

$$\Phi(\rho, t) = -\frac{1}{2}\rho\frac{\mathrm{d}\zeta}{\mathrm{d}t} + \phi_0(t)$$

where $\phi_0(t)$ is a function to be determined. Next we eliminate ρ from the real part of (3.5) ($\rho = x - \zeta$), consequently $\zeta = \zeta_0$, a constant, which we set equal to zero. Thus (3.4) simplifies to

$$P'' + (b_1 + \mathrm{i}b_2)(P^2Q' + PQP') + (a_1 + \mathrm{i}a_2)(P^2Q' + 2PQP') + cP^3Q^2 + z^{-1}P' + dP^2Q - \left(\frac{\mathrm{d}\phi_0}{\mathrm{d}t} + \frac{\kappa^2}{z^2}\right)P = 0.$$

From the coefficient of P we see that $\mathrm{d}\phi_0/\mathrm{d}t$ must also be a constant and so $\phi_0(t) = \lambda t + \mu$, with λ and μ arbitrary constants. Hence we obtain the reduction

$$u(\rho, t) = P(z)\exp\{\mathrm{i}(\lambda t + \mu)\} \tag{3.6a}$$

$$v(\rho, t) = Q(z)\exp\{-\mathrm{i}(\lambda t + \mu)\} \tag{3.6b}$$

with $z = \rho$, and where P and Q satisfy

$$P'' + (b_1 + \mathrm{i}b_2)(P^2Q' + PQP') + (a_1 + \mathrm{i}a_2)(P^2Q' + 2PQP') + cP^3Q^2 + dP^2Q + z^{-1}P' - (\lambda + \kappa^2/z^2)P = 0 \tag{3.7a}$$

$$Q'' + (b_1 - \mathrm{i}b_2)(Q^2P' + PQQ') + (a_1 - \mathrm{i}a_2)(Q^2P' + 2PQQ') + cP^2Q^3 + dPQ^2 + z^{-1}Q' - (\lambda + \kappa^2/z^2)Q = 0. \tag{3.7b}$$

Setting $P(z) = R(z)\exp\{\mathrm{i}\Theta(z)\}$ and $Q(z) = R(z)\exp\{-\mathrm{i}\Theta(z)\}$ in these equations yields

$$R'' - R(\Theta')^2 + (2b_1 + 3a_1)R^2R' - a_2R^3\Theta' + z^{-1}R' + cR^5 + dR^3 - (\lambda + \kappa^2/z^2)R = 0 \tag{3.8a}$$

$$R\Theta'' + 2R'\Theta' + z^{-1}R\Theta' + a_1R^3\Theta' + (2b_2 + 3a_2)R^2R' = 0. \tag{3.8b}$$

If $a_1 = 0$ and $a_2 = -\frac{2}{3}b_2$, then (3.8b) can be integrated to give $zR^2(z)\Theta'(z) = \gamma$, with γ an arbitrary constant, then eliminating Θ' in (3.8a) yields

$$R'' + \frac{R'}{z} + 2b_1R^2R' + cR^5 + dR^3 - \left(\lambda - \frac{2\gamma b_2}{3z} + \frac{\kappa^2}{z^2}\right)R - \frac{\gamma^2}{z^2R^3} = 0. \tag{3.9}$$

In section 4, we obtain solutions of this equation expressed in terms of the fourth Painlevé equation (if $b_1 = 0$, $\lambda = -3d^2/(16c)$ and $\kappa^2 = \frac{1}{16}$, with $d \neq 0$), Jacobi elliptic functions (if $b_1 = d = \lambda = 0$ and $\kappa^2 = \frac{1}{16}$).

3.2. $\beta^2(t) = \eta(t)$, *with* $a_1 = a_2 = d = 0$

In this case, (1.1) reduces to

$$\mathrm{i}u_t + u_{\rho\rho} + \rho^{-1}u_\rho - \kappa^2\rho^{-2}u + (b_1 + \mathrm{i}b_2)u(|u|^2)_\rho + cu|u|^4 = 0 \tag{3.10}$$

and (3.3) reduces to

$$\eta^3P'' + (b_1 + \mathrm{i}b_2)\eta^3(P^2Q' + PQP') + \left[\frac{\eta^2}{\rho} + \mathrm{i}\left(\rho\eta\frac{\mathrm{d}\eta}{\mathrm{d}t} + 2\Phi_\rho\eta^2 + \frac{\mathrm{d}\zeta}{\mathrm{d}t}\eta\right)\right]P'$$
$$+c\eta^3P^3Q^2 + \left[\mathrm{i}\left(\frac{\eta}{\rho}\Phi_\rho + \tfrac{1}{2}\frac{\mathrm{d}\eta}{\mathrm{d}t} + \eta\Phi_{\rho\rho}\right) - \eta\left(\Phi_t + \Phi_\rho^2 + \frac{\kappa^2}{\rho^2}\right)\right]P = 0. \tag{3.11}$$

First, consider the coefficient of P'; comparing this with the coefficient of P'' shows that, for (3.11) to be an ODE, then necessarily

$$\mathrm{i}\rho\eta\frac{\mathrm{d}\eta}{\mathrm{d}t} + \frac{\eta^2}{\rho} + 2\mathrm{i}\Phi_\rho\eta^2 + \mathrm{i}\frac{\mathrm{d}\zeta}{\mathrm{d}t}\eta = \eta^3[\Gamma_{21}(z) + \mathrm{i}\Gamma_{22}(z)] \tag{3.12}$$

where $\Gamma_{21}(z)$ and $\Gamma_{22}(z)$ are functions to be determined. The real part of (3.12) implies that $\zeta \equiv 0$ ($\Gamma_{21} \equiv 1/z$), and using this in the imaginary part of (3.12) (set $\Gamma_{22}(z) \equiv 0$, without loss of generality) then, since Φ is the only function which depends on ρ, we obtain

$$\Phi(\rho, t) = -\frac{\rho^2}{4\eta}\frac{\mathrm{d}\eta}{\mathrm{d}t} + \phi_0(t) \tag{3.13}$$

where $\phi_0(t)$ is to be determined. Substituting (3.13) into (3.11) and eliminating $\rho(= z/\eta)$ yields

$$P'' + (b_1 + \mathrm{i}b_2)(PQP' + P^2Q') + cP^3Q^2 + z^{-1}P'$$
$$+ \left\{\frac{1}{4}z^2\left[\frac{1}{\eta^5}\frac{\mathrm{d}^2\eta}{\mathrm{d}t^2} - \frac{2}{\eta^6}\left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\right)^2\right] - \frac{1}{\eta^2}\frac{\mathrm{d}\phi_0}{\mathrm{d}t} - \frac{\alpha}{z^2} - \frac{\mathrm{i}}{2\eta^3}\frac{\mathrm{d}\eta}{\mathrm{d}t}\right\}P = 0.$$

Separating the real and imaginary parts of the coefficient of P in this gives

$$\frac{1}{4}z^2\left[\frac{1}{\eta^5}\frac{\mathrm{d}^2\eta}{\mathrm{d}t^2} - \frac{2}{\eta^6}\left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\right)^2\right] - \frac{1}{\eta^2}\frac{\mathrm{d}\phi_0}{\mathrm{d}t} - \frac{\alpha}{z^2} = \Gamma_{31}(z) \tag{3.14a}$$

$$-\frac{1}{2\eta^3}\frac{\mathrm{d}\eta}{\mathrm{d}t} = \Gamma_{32}(z). \tag{3.14b}$$

where $\Gamma_{31}(z)$ and $\Gamma_{32}(z)$ are functions to be determined. Since η depends only on t, then $\Gamma_{32} = \gamma$, a constant (set $\gamma = 1$, without loss of generality) and so

$$\eta(t) = t^{-1/2}. \tag{3.15}$$

The left-hand side of (3.14*a*) implies that $\Gamma_{31}(z) = \delta z^2 + \mu - \kappa^2/z^2$, with δ and μ constants. Substituting this and (3.15) into (3.14*b*) yields

$$\frac{1-16\delta}{16t} = 0 \qquad -t\frac{\mathrm{d}\phi_0}{\mathrm{d}t} - \mu = 0$$

and consequently $\delta = 1/16$ and $\phi_0(t) = -\mu \ln t$.

Thus we have obtained the reduction

$$u(\rho,t) = t^{-1/4}P(z)\exp\{\mathrm{i}(\tfrac{1}{8}z^2 - \mu\ln t)\} \tag{3.16a}$$

$$v(\rho,t) = t^{-1/4}Q(z)\exp\{-\mathrm{i}(\tfrac{1}{8}z^2 - \mu\ln t)\} \tag{3.16b}$$

with $z = \rho/t^{1/2}$, and where $P(z)$ and $Q(z)$ satisfy

$$P'' + (b_1 + \mathrm{i}b_2)(PQP' + P^2Q') + cP^3Q^2 + z^{-1}P' + (\tfrac{1}{16}z^2 + \tfrac{1}{4}\mathrm{i} + \mu - \kappa^2/z^2)P = 0 \tag{3.17a}$$

$$Q'' + (b_1 - \mathrm{i}b_2)(PQQ' + Q^2P') + cP^2Q^3 + z^{-1}Q' + (\tfrac{1}{16}z^2 - \tfrac{1}{4}\mathrm{i} + \mu - \kappa^2/z^2)Q = 0. \tag{3.17b}$$

Setting $P(z) = R(z)\exp\{\mathrm{i}\Theta(z)\}$ and $Q(z) = R(z)\exp\{-\mathrm{i}\Theta(z)\}$ in these equations yields

$$R'' - R(\Theta')^2 + 2b_1R^2R' + z^{-1}R' + cR^5 + (\tfrac{1}{16}z^2 + \mu - \kappa^2/z^2)R = 0 \tag{3.18a}$$

$$R\Theta'' + 2R'\Theta' + z^{-1}R\Theta' + 2b_2R^2R' + \tfrac{1}{4}R = 0 \tag{3.18b}$$

If $b_2 = 0$, then (3.18*b*) can be integrated to give $zR^2(z)[\Theta'(z) + \frac{1}{8}z] = \gamma$, with γ an arbitrary constant, then eliminating Θ' in (3.18*a*) yields

$$R'' + \frac{R'}{z} + 2b_1R^2R' + cR^5 + \left(\frac{3z^2}{64} + \mu - \frac{\kappa^2}{z^2}\right)R + \frac{\gamma}{4R} - \frac{\gamma^2}{z^2R^3} = 0. \tag{3.19}$$

3.3. $\beta(t) = \eta(t)$ *with* $a_1 = a_2 = b_1 = b_2 = c = 0$

In this case, (1.1) reduces to

$$\mathrm{i}u_t + u_{\rho\rho} + \rho^{-1}u_\rho - \kappa^2\rho^{-2}u + cu|u|^2 = 0 \tag{3.20}$$

which is the special case of the $(2+1)$-dimensional cubic nonlinear Schrödinger equation (2.5) when $\psi(x,y,t) = u(\rho,t)\exp\{\mathrm{i}\kappa\theta\}$, with $\rho = (x^2+y^2)^{1/2}$ and $\theta = \tan^{-1}(y/x)$, and (3.3) reduces to

$$\eta^3P'' + d\eta^3P^2Q + \left[\frac{\eta^2}{\rho} + \mathrm{i}\eta\left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\rho + 2\eta\Phi_\rho + \frac{\mathrm{d}\zeta}{\mathrm{d}t}\right)\right]P' + \left[\mathrm{i}\left(\frac{\eta\Phi_\rho}{\rho} + \frac{\mathrm{d}\eta}{\mathrm{d}t} + \eta\Phi_{\rho\rho}\right) - \eta\left(\Phi_t + \Phi_\rho^2 + \frac{\kappa^2}{\rho^2}\right)\right]P = 0. \tag{3.21}$$

We remark that this equation has a similar form to (3.11), in particular, the coefficient of P is only changed by a factor of 2 in the second term; however, significantly, the coefficient of P' has only an imaginary part. This means that the form for $\Phi(\rho,t)$ will be unchanged from section 3.2, and is given by (3.13), but there is one less equation determining $\eta(t)$ which turns out to be crucial.

Proceeding in an analogous way to the previous section, we find that ζ must again be a constant and, as before, we set it to zero. Eliminating $\rho(=z/\eta)$ in (3.21) yields

$$P'' + dP^2Q + \frac{P'}{z} + \left\{\left[\frac{1}{4\eta^5}\frac{\mathrm{d}^2\eta}{\mathrm{d}t^2} - \frac{1}{2\eta^6}\left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\right)^2\right] z^2 - \frac{\kappa^2}{z^2} - \mathrm{i}\left(\frac{1}{\eta^2}\frac{\mathrm{d}\phi_0}{\mathrm{d}t}\right)\right\} P = 0. \tag{3.22}$$

The coefficients of z^2P and P are constants, say δ and μ, respectively, thus

$$\left[\frac{1}{4\eta^5}\frac{\mathrm{d}^2\eta}{\mathrm{d}t^2} - \frac{1}{2\eta^6}\left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\right)^2\right] = \delta \qquad -\frac{\mathrm{d}\phi_0}{\mathrm{d}t} = \mu\eta^2. \tag{3.23}$$

There are three sets canonical solutions for these equations:

$$\eta(t) = 1/t^{1/2} \qquad \phi_0(t) = -\mu \ln t \tag{3.24a}$$

$$\eta(t) = 1/t \qquad \phi_0(t) = \mu/t \tag{3.24b}$$

$$\eta(t) = 1/(t^2+1)^{1/2} \qquad \phi_0(t) = -\mu \tan^{-1} t \tag{3.24c}$$

for $\delta = \frac{1}{16}$, $\delta = 0$ and $\delta = -\frac{1}{4}$, respectively. Hence we obtain the following three reductions:

$$u(\rho,t) = t^{-1/2}P(z)\exp\{\mathrm{i}(\tfrac{1}{8}z^2 - \mu\ln t)\} \tag{3.25a}$$

$$v(\rho,t) = t^{-1/2}Q(z)\exp\{-\mathrm{i}(\tfrac{1}{8}z^2 - \mu\ln t)\} \tag{3.25b}$$

with $z = \rho/t^{1/2}$;

$$u(\rho,t) = t^{-1}P(z)\exp\{\mathrm{i}(\tfrac{1}{4}\rho^2 + \mu)/t\} \tag{3.26a}$$

$$v(\rho,t) = t^{-1}Q(z)\exp\{-\mathrm{i}(\tfrac{1}{4}\rho^2 + \mu)/t\} \tag{3.26b}$$

with $z = \rho/t$; and

$$u(\rho,t) = (t^2+1)^{-1/2}P(z)\exp\left\{\mathrm{i}\left[\frac{\rho^2 t}{4(t^2+1)} - \mu\tan^{-1}t\right]\right\} \tag{3.27a}$$

$$v(\rho,t) = (t^2+1)^{-1/2}Q(z)\exp\left\{-\mathrm{i}\left[\frac{\rho^2 t}{4(t^2+1)} - \mu\tan^{-1}t\right]\right\} \tag{3.27b}$$

with $z = \rho/(t^2+1)^{1/2}$. For each reduction $P(z)$ and $Q(z)$ satisfy

$$P'' + z^{-1}P' + dP^2Q + (\delta z^2 + \mu - \kappa^2/z^2)P = 0 \tag{3.28a}$$

$$Q'' + z^{-1}Q' + dPQ^2 + (\delta z^2 + \mu - \kappa^2/z^2)Q = 0 \tag{3.28b}$$

with $\delta = \frac{1}{16}$, $\delta = 0$ and $\delta = -\frac{1}{4}$, respectively. Setting $P(z) = R(z)\exp\{\mathrm{i}\Theta(z)\}$ and $Q(z) = R(z)\exp\{-\mathrm{i}\Theta(z)\}$ in these equations yields

$$R'' - R(\Theta')^2 + z^{-1}R' + dR^3 + (\delta z^2 + \mu - \kappa^2/z^2)R = 0 \qquad (3.29a)$$

$$R\Theta'' + 2R'\Theta' + z^{-1}R\Theta' = 0. \qquad (3.29b)$$

Equation (3.29*b*) can be integrated to give $zR^2(z)\Theta'(z) = \gamma$, with γ an arbitrary constant, then eliminating Θ' in (3.29*a*) yields

$$R'' + \frac{R'}{z} + dR^3 + \left(\delta z^2 + \mu - \frac{\kappa^2}{z^2}\right)R - \frac{\gamma^2}{z^2R^3} = 0. \qquad (3.30)$$

In section 4, we obtain solutions of this equation expressed in terms of the second Painlevé equation (if $\mu \neq 0$, $\delta = 0$ and $\kappa^2 = \frac{1}{9}$), and elliptic functions (if $\mu = 0$, $\delta = 0$ and $\kappa^2 = \frac{1}{9}$).

Hence we have shown that (3.20), a special case of the $(2+1)$-dimensional cubic nonlinear Schrödinger equation (2.5), has two additional symmetry reductions, i.e. (3.26) and (3.27), in comparison with (3.10), which has a quintic nonlinearity. These additional symmetry reductions are generated by the vector field

$$\boldsymbol{v} = \rho t\partial_\rho + (t^2+\delta)\partial_t - (t + \mathrm{i}\mu - \tfrac{1}{4}\mathrm{i}\rho^2)u\partial_u - (t - \mathrm{i}\mu + \tfrac{1}{4}\mathrm{i}\rho^2)v\partial_v \qquad (3.31)$$

with $\delta = 0$ for (3.26) and $\delta = \frac{1}{4}$ for (3.27). In the special case $\delta = \mu = 0$, the vector field (3.31) reduces to

$$\boldsymbol{v} = \rho t\partial_\rho + t^2\partial_t - (t - \tfrac{1}{4}\mathrm{i}\rho^2)u\partial_u - (t + \tfrac{1}{4}\mathrm{i}\rho^2)v\partial_v \qquad (3.32)$$

which represents a conformal point symmetry and the associated one-parameter transformation group is

$$\hat{\rho} = \frac{\rho}{1-\epsilon t} \qquad \hat{u} = u(1-\epsilon t)\exp\left\{\frac{\mathrm{i}\epsilon\rho^2}{4(1-\epsilon t)}\right\} \qquad (3.33a)$$

$$\hat{t} = \frac{t}{1-\epsilon t} \qquad \hat{v} = v(1-\epsilon t)\exp\left\{-\frac{\mathrm{i}\epsilon\rho^2}{4(1-\epsilon t)}\right\} \qquad (3.33b)$$

which is the *Talanov lens transformation* (Talanov 1970). This transformation can be interpreted as the image of a field $\hat{u}(\hat{\rho},\hat{t})$ produced by a thin optical lens with focal length $1/\epsilon$.

4. Painlevé analysis of ODES resulting from the GCNLS equation

In this section we apply Painlevé analysis to the following equations

$$R'' + \frac{R'}{z} + dR^3 + \left(\delta z^2 + \mu - \frac{\kappa^2}{z^2}\right)R - \frac{\gamma^2}{z^2R^3} = 0 \qquad (4.1)$$

$$R'' + \frac{R'}{z} + 2b_1R^2R' + cR^5 + \left(\frac{3z^2}{64} + \mu - \frac{\kappa^2}{z^2}\right)R + \frac{\gamma}{4R} - \frac{\gamma^2}{z^2R^3} = 0 \qquad (4.2)$$

$$R'' + \frac{R'}{z} + 2b_1R^2R' + cR^5 + dR^3 - \left(\lambda - \frac{2\gamma b_2}{3z} + \frac{\kappa^2}{z^2}\right)R - \frac{\gamma^2}{z^2R^3} = 0. \qquad (4.3)$$

4.1. Equation (4.1)

Making the transformation $R(z) = w^{1/2}(z)$ in (4.1) yields

$$ww'' - \frac{1}{2}(w')^2 + \frac{ww'}{z} + 2dw^3 + 2\left(\delta z^2 + \mu - \frac{\kappa^2}{z^2}\right) w^2 - \frac{2\gamma^2}{z^2} = 0. \qquad (4.4)$$

It is easily shown using the algorithm of Ablowitz *et al* (1980) that this is of Painlevé type if and only if $\delta = 0$ and $\kappa^2 = \frac{1}{9}$, i.e. for the special case of (4.4) given by

$$ww'' - \frac{1}{2}(w')^2 + \frac{ww'}{z} + 2dw^3 + 2\left(\mu - \frac{1}{9z^2}\right) w^2 - \frac{2\gamma^2}{z^2} = 0. \qquad (4.5)$$

Next we transform (4.5) into so-called standard form to allow us to find its solutions in terms of the classification of Painlevé and Gambier (Ince 1956). Following the procedure described by Ince, we make the transformation

$$w(z) = \chi(z)W(Z) \qquad Z(z) = \psi_0 z^{2/3} \qquad \chi(z) = -\frac{8\psi_0^2}{9d} z^{-2/3} \qquad (4.6)$$

where ψ_0 is a constant, which yields

$$\frac{d^2W}{dZ^2} = \frac{1}{2W}\left(\frac{dW}{dZ}\right)^2 + 4W^2 + \frac{3^6\gamma^2 d^2}{2^7\psi_0^6}\frac{1}{W} - \frac{9\mu}{2\psi_0^3}ZW. \qquad (4.7)$$

There are four cases to consider:

Case 4.1(i). If $\mu = \gamma = 0$ then (4.7) reduces to equation (XVIII) in chapter 14 of Ince (1956). Its first integral is

$$(dW/dZ)^2 = 4W(W_0 + W^2)$$

where W_0 is the constant of integration, which is solvable in terms of elliptic functions (if $W_0 \neq 0$), or elementary functions (if $W_0 = 0$).

Case 4.1(ii). If $\mu = 0$ and $\gamma \neq 0$ then we set $\psi_0 = \frac{3}{2}\mathrm{i}(\gamma d)^{1/3}$, and obtain a special case of equation (XXXIII) in chapter 14 of Ince (1956). Its first integral is

$$(dW/dZ)^2 = 4W^3 + 4W_0W + 1$$

where W_0 is the constant of integration, which is also solvable in terms of elliptic functions (if $W_0 \neq 0$), or elementary functions (if $W_0 = 0$).

Case 4.1(iii). If $\mu \neq 0$ and $\gamma = 0$, then we set $\psi_0 = (-\frac{9}{4}\mu)^{1/3}$ and obtain equation (XX) in chapter 14 of Ince (1956)

$$\frac{d^2W}{dZ^2} = \frac{1}{2W}\left(\frac{dW}{dZ}\right)^2 + 4W^2 + 2ZW.$$

Setting $W = Y^2$ yields

$$\frac{d^2Y}{dZ^2} = 2Y^3 + ZY$$

which is a special case of the second Painlevé equation (P_{II}).

Case 4.1(iv). Finally, with $\mu \neq 0$ and $\gamma \neq 0$, we set $\psi_0 = (\frac{9}{2}\mu)^{1/3}$, and after rescaling W we obtain equation (XXXIV) in chapter 14 of Ince (1956)

$$\frac{\mathrm{d}^2 W}{\mathrm{d}Z^2} = \frac{1}{2W}\left(\frac{\mathrm{d}W}{\mathrm{d}Z}\right)^2 + 4\alpha W^2 - ZW - \frac{1}{2W}$$

with $\alpha = \pm 3\mathrm{i}\gamma d/4mu$. This equation has the solution

$$W = \frac{1}{2\alpha}\left(\frac{\mathrm{d}V}{\mathrm{d}Z} + V^2 + \tfrac{1}{2}Z\right)$$

where $V(Z)$ satisfies $\mathrm{P_{II}}$

$$\mathrm{d}^2 V/\mathrm{d}Z^2 = 2V^3 + ZV - 2\alpha - \tfrac{1}{2}.$$

For further details of exact solutions of (4.7) see Gagnon and Winternitz (1989c).

4.2. *Equation (4.2)*

Making the transformation $R(z) = w^{1/2}(z)$ in (4.2) yields

$$ww'' - \tfrac{1}{2}(w')^2 + \frac{ww'}{z} + 2b_1 w^2 w' + 2cw^4 + 2\left(\frac{3z^2}{64} + \mu - \frac{\kappa^2}{z^2}\right)w^2 + \tfrac{1}{2}\gamma w - \frac{2\gamma^2}{z^2} = 0. \tag{4.8}$$

Using the algorithm of Ablowitz *et al* (1980), it is easily shown that this equation is not of Painlevé type (if either $b_1 \neq 0$ or $c \neq 0$).

4.3. *Equation (4.3)*

Making the transformation $R(z) = w^{1/2}(z)$ in (4.3) yields

$$ww'' - \tfrac{1}{2}(w')^2 + \frac{ww'}{z} + 2b_1 w^2 w' + 2cw^4 + 2dw^3 + 2\left(\lambda - \frac{2b_2\gamma}{3z} + \frac{\kappa^2}{z^2}\right)w^2 - \frac{2\gamma^2}{z^2} = 0. \tag{4.9}$$

There are three cases to consider

Case 4.3(i). $b_1 \neq 0$. In this case it is easily shown that (4.9) is not of Painlevé type.

Case 4.3(ii). $b_1 = 0$ and $c \neq 0$. In this case it is easily shown that (4.9) is of Painlevé type if and only if

$$\lambda = -\frac{3d^2}{16c} \qquad \kappa^2 = \frac{1}{16}$$

i.e. for the special case given by

$$ww'' - \tfrac{1}{2}(w')^2 + \frac{ww'}{z} + 2cw^4 + 2dw^3 + 2\left(-\frac{3d^2}{16c} - \frac{2b_2\gamma}{3z} + \frac{1}{16z^2}\right)w^2 - \frac{2\gamma^2}{z^2} = 0. \tag{4.10}$$

As for (4.5), we can transform (4.10) into standard form and thus solve it.

If $d \neq 0$, then making the transformation

$$w(z) = \chi(z)W(Z) \qquad Z(z) = \left(\frac{3d^2}{4(-c)}\right)^{1/4} z^{1/2}$$
$$\chi(z) = \left(\frac{3^3 d^2}{2^{10}(-c)^3}\right)^{1/4} z^{-1/2} \tag{4.11}$$

to (4.10) yields the fourth Painlevé equation ($\mathrm{P_{IV}}$)

$$\frac{\mathrm{d}^2 W}{\mathrm{d}Z^2} = \frac{1}{2W}\left(\frac{\mathrm{d}W}{\mathrm{d}Z}\right)^2 + \frac{3}{2}W^3 + 4ZW^2 + 2W(Z^2 - \alpha) + \frac{\beta}{W}$$

with

$$\alpha = \frac{16 b_2 \gamma (-c)^{1/2}}{3^{3/2} d} \qquad \beta = \frac{2^9 c^2 \gamma^2}{9 d^2}.$$

If $d = 0$, then making the transformation

$$w(z) = z^{-1/2} W(Z) \qquad Z(z) = 4\left(-\tfrac{1}{3}c\right)^{1/2} z^{1/2} \tag{4.12}$$

to (4.10) yields

$$\frac{\mathrm{d}^2 W}{\mathrm{d}Z^2} = \frac{1}{2W}\left(\frac{\mathrm{d}W}{\mathrm{d}Z}\right)^2 + \frac{3}{2}W^3 - \alpha W + \frac{\beta}{W} \tag{4.13}$$

with $\alpha = b_2\gamma/c$ and $\beta = -3\gamma^2/(2c)$, which is solvable in terms of Jacobian elliptic functions.

We remark that Gagnon and Paré (1990) obtained approximate solutions of (4.9) with $b_1 = 0$ using a variational technique and compared the results with numerical simulations.

Case 4.3(iii). $b_1 = 0$ and $c = 0$. In this case it is easily shown that (4.9) is of Painlevé type if and only if $d = 0$, in when it simplifies to

$$ww'' - \tfrac{1}{2}(w')^2 + \frac{ww'}{z} + 2\left(\lambda - \frac{2b_2\gamma}{3z} + \frac{\kappa^2}{z^2}\right) w^2 - \frac{2\gamma^2}{z^2} = 0. \tag{4.14}$$

Making the transformation $w(z) = z^{-1} y(x)$, $x = 2\lambda^{1/2} z$, yields the Pinney equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \left(-\frac{1}{4} + \frac{b_2\gamma}{3\lambda^{1/2} x} + \frac{\frac{1}{4} - \kappa^2}{x^2}\right) y = \frac{\gamma^2}{4\lambda y^3}. \tag{4.15}$$

The general solution of (4.15) is given by

$$y(x) = \left(y_1^2 + \frac{\gamma^2 y_2^2}{4\lambda}\right)^{1/2} \tag{4.16}$$

where $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the Whittaker equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \left(-\frac{1}{4} + \frac{b_2\gamma}{3\lambda^{1/2} x} + \frac{\frac{1}{4} - \kappa^2}{x^2}\right) y = 0, \tag{4.17a}$$

satisfying

$$y_1 \frac{\mathrm{d}y_2}{\mathrm{d}x} - y_2 \frac{\mathrm{d}y_1}{\mathrm{d}x} = 1 \tag{4.17b}$$

(Pinney 1950).

5. Coupled, cylindrical nonlinear Schrödinger equation

In this section we discuss symmetry reductions of the $(2+1)$-dimensional coupled nonlinear Schrödinger equations in cylindrical coordinates

$$\mathrm{i}\psi_{1,t} + \psi_{1,\rho\rho} + \rho^{-1}\psi_{1,\rho} + \rho^{-2}\psi_{1,\theta\theta} + (\alpha|\psi_1|^2 + \beta|\psi_2|^2)\psi_1 = 0 \tag{5.1a}$$

$$\mathrm{i}\psi_{2,t} + \psi_{2,\rho\rho} + \rho^{-1}\psi_{2,\rho} + \rho^{-2}\psi_{2,\theta\theta} + (\gamma|\psi_1|^2 + \delta|\psi_2|^2)\psi_2 = 0 \tag{5.1b}$$

with α, β, γ and δ arbitrary constants (not all zero). Equation (5.1) is of particular interest in the field of transverse effects in nonlinear optics (cf Abraham and Firth 1990).

The application of the direct method to equations (5.1) is very similar to that for (1.1) in section 3, and so we omit details. The method yields the following four symmetry reductions

Example 5.1. The time-translational symmetry reduction

$$\psi_1(\rho,\theta,t) = P_1(z)\exp\{\mathrm{i}(\kappa\theta + \lambda t)\} \tag{5.2a}$$

$$\psi_2(\rho,\theta,t) = P_2(z)\exp\{\mathrm{i}(\kappa\theta + \lambda t)\} \tag{5.2b}$$

with $z = \rho$, and where $P_1(z)$ and $P_2(z)$ satisfy

$$P_1'' + z^{-1}P_1' - (\lambda + \kappa^2/z^2)P_1 + (\alpha|P_1|^2 + \beta|P_2|^2)P_1 = 0 \tag{5.3a}$$

$$P_2'' + z^{-1}P_2' - (\lambda + \kappa^2/z^2)P_2 + (\gamma|P_1|^2 + \delta|P_2|^2)P_2 = 0. \tag{5.3b}$$

It can be shown that these equations are not of Painlevé-type (unless $\alpha = \beta = \gamma = \delta = 0$).

Example 5.2. The scaling (or dilational) symmetry reduction

$$\psi_1(\rho,\theta,t) = t^{-1/2}P_1(z)\exp\{\mathrm{i}[\kappa\theta + (\tfrac{1}{8}z^2 - \mu\ln t)]\} \tag{5.4a}$$

$$\psi_2(\rho,\theta,t) = t^{-1/2}P_2(z)\exp\{\mathrm{i}[\kappa\theta + (\tfrac{1}{8}z^2 - \mu\ln t)]\} \tag{5.4b}$$

with $z = \rho/t^{1/2}$.

Example 5.3. The conformal point symmetry reduction

$$\psi_1(\rho,\theta,t) = t^{-1}P_1(z)\exp\{\mathrm{i}[\kappa\theta + (\tfrac{1}{4}\rho^2 + \mu)/t]\} \tag{5.5a}$$

$$\psi_2(\rho,\theta,t) = t^{-1}P_2(z)\exp\{\mathrm{i}[\kappa\theta + (\tfrac{1}{4}\rho^2 + \mu)/t]\} \tag{5.5b}$$

with $z = \rho/t$.

Example 5.4. The generalized conformal point symmetry reduction

$$\psi_1(\rho,\theta,t) = (t^2+1)^{-1/2} P_1(z) \exp\left\{ \mathrm{i}\left[\kappa\theta + \frac{\rho^2 t}{4(t^2+1)} - \mu \tan^{-1} t\right]\right\} \qquad (5.6a)$$

$$\psi_2(\rho,\theta,t) = (t^2+1)^{-1/2} P_2(z) \exp\left\{ \mathrm{i}\left[\kappa\theta + \frac{\rho^2 t}{4(t^2+1)} - \mu \tan^{-1} t\right]\right\} \qquad (5.6b)$$

with $z = \rho/(t^2+1)^{1/2}$.

For the reduction in examples 5.2, 5.3 and 5.4, $P_1(z)$ and $P_2(z)$ satisfy

$$P_1'' + z^{-1}P_1' + (\delta z^2 + \mu - \kappa^2/z^2)P_1 + (\alpha|P_1|^2 + \beta|P_2|^2)P_1 = 0 \qquad (5.7a)$$

$$P_2'' + z^{-1}P_2' + (\delta z^2 + \mu - \kappa^2/z^2)P_2 + (\gamma|P_1|^2 + \delta|P_2|^2)P_2 = 0 \qquad (5.7b)$$

with $\delta = \frac{1}{16}$, $\delta = 0$ and $\delta = -\frac{1}{4}$, respectively. It can be shown that these equations are not of Painlevé-type (unless $\alpha = \beta = \gamma = \delta = 0$).

Gagnon (1992) has recently discussed the (classical) Lie symmetries of the special case of

$$\mathrm{i}\psi_{1,t} + \psi_{1,\rho\rho} + \rho^{-1}\psi_{1,\rho} + \rho^{-2}\psi_{1,\theta\theta} + \sigma[|\psi_1|^2 + (1+h)|\psi_2|^2]\psi_1 = 0 \qquad (5.8a)$$

$$\mathrm{i}\psi_{2,t} + \psi_{2,\rho\rho} + \rho^{-1}\psi_{2,\rho} + \rho^{-2}\psi_{2,\theta\theta} + \sigma[(1+h)|\psi_1|^2 + |\psi_2|^2]\psi_2 = 0 \qquad (5.8b)$$

where $\sigma = \pm 1$, and obtained some exact and approximate solutions. In particular, Gagnon (1992) studied exact approximate solutions arising from the generalized conformal point symmetry reduction (5.6) by solving approximately (5.7) with $\alpha = \delta = \pm 1$ and $\beta = \gamma = \pm(1+h)$, where h is a non-zero real parameter, using variational techniques. These approximate solutions were expressed in terms of Laguerre–Gauss polynomials and generalized earlier results due to Marburger and Felber (1978).

6. The n-dimensional nonlinear Schrödinger equation

In section 3 we obtained a conformal point symmetry reduction of (3.1) in the form

$$u(\rho,t) = t^\beta P(z)\exp\{\mathrm{i}(\alpha\rho^2+\mu)/t\} \qquad (6.1a)$$

$$v(\rho,t) = t^\beta Q(z)\exp\{-\mathrm{i}(\alpha\rho^2+\mu)/t\} \qquad (6.1b)$$

with $z = \rho/t$, and where α and β are specified constants and μ is an arbitrary constant. Here, we investigate whether analogous reductions arise for the $(n+1)$-dimensional nonlinear Schrödinger equation with radial symmetry

$$\mathrm{i}u_t + \nabla^2 u + c|u|^{2\sigma}u = 0 \qquad (6.2a)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial\rho^2} + \frac{n-1}{\rho}\frac{\partial}{\partial\rho}. \qquad (6.2b)$$

This equation arises in various physical context describing the slowly varying envelope wave train in conservative dispersive systems. It is a generic equation describing the slowly varying envelope wave train in conservative dispersive systems. For $c < 0$ and $\sigma < 2/(n-2)$, if $n > 2$ or $\sigma < \infty$, if $n \leqslant 2$, then it has solitary wave solutions which are stable for $\sigma < 2/n$ and for $\sigma \geqslant 2/n$, there exist solutions of (6.2) which blow up at a finite time (cf LeMesurier *et al* 1988a, Rasmussen and Rypdal 1986). Reductions of (6.2) have been considered by Gagnon and Winternitz (1990) in cylindrical ($n = 1$) and spherical ($n = 2$) coordinates.

As in section 3, we rewrite (6.1) as the system

$$\mathrm{i}u_t + \nabla^2 u + cu(uv)^\sigma = 0 \tag{6.3a}$$

$$-\mathrm{i}v_t + \nabla^2 v + cv(uv)^\sigma = 0. \tag{6.3b}$$

Substituting (6.1) into (6.6), multiplying through by $t^{2-\beta}\exp\{-\mathrm{i}(\alpha\rho^2 + \mu)/t\}$ and eliminating ρ yields

$$P'' + (4\alpha - 1)\mathrm{i}tzP' + (n-1)z^{-1}P' + [(\alpha - 4\alpha^2)t^2z^2 + \mu + \mathrm{i}(2\alpha n + \beta)t]P + ct^{2(\sigma\beta+1)}P(PQ)^\sigma = 0. \tag{6.4a}$$

$$Q'' - (4\alpha - 1)\mathrm{i}tzQ' + (n-1)z^{-1}Q' + [\alpha(1 - 4\alpha)t^2z^2 + \mu + \mathrm{i}(2\alpha n + \beta)t]Q + ct^{2(\sigma\beta+1)}Q(PQ)^\sigma = 0. \tag{6.4b}$$

In order for these to be ODES, then necessarily

$$\alpha = \tfrac{1}{4} \qquad \sigma\beta + 1 = 0 \qquad 2\alpha n + \beta = 0. \tag{6.5}$$

Hence we see that (6.3) possesses a symmetry reduction of the form (6.1) with $\alpha = \frac{1}{4}$ and $\beta = -1/\sigma$, provided that

$$\sigma n = 2 \tag{6.6}$$

which is the so-called critical dimension, and where $P(z)$ and $Q(z)$ satisfy

$$P'' + \left(\frac{2}{\sigma} - 1\right)\frac{P'}{z} + \mu P + cP(PQ)^\sigma = 0. \tag{6.7a}$$

$$Q'' + \left(\frac{2}{\sigma} - 1\right)\frac{Q'}{z} + \mu Q + cQ(PQ)^\sigma = 0. \tag{6.7b}$$

It is easily shown that these equations are not of Painlevé type.

Similarly it can be shown that (6.3) possesses the generalized conformal point symmetry reduction

$$u(\rho, t) = (t^2 + 1)^{-1/(2\sigma)}P(z)\exp\left\{\mathrm{i}\left[\frac{\rho^2 t}{4(t^2+1)} - \mu\tan^{-1}t\right]\right\} \tag{6.8a}$$

$$v(\rho, t) = (t^2 + 1)^{-1/(2\sigma)}Q(z)\exp\left\{-\mathrm{i}\left[\frac{\rho^2 t}{4(t^2+1)} - \mu\tan^{-1}t\right]\right\} \tag{6.8b}$$

with $z = \rho/(t^2 + 1)^{1/2}$, if and only (6.6) holds. It is easily shown that these equations are not of Painlevé type.

We remark that for all n and σ, (6.3) also possesses the time-translational symmetry reduction

$$u(\rho,t) = P(z)\exp(\mathrm{i}\mu t) \tag{6.9a}$$

$$v(\rho,t) = Q(z)\exp(-\mathrm{i}\mu t) \tag{6.9b}$$

where $z=\rho$ and $P(z)$ and $Q(z)$ satisfy

$$P'' + (n-1)z^{-1}P' + [\mu - \tfrac{1}{4}z^2]P + cP(PQ)^\sigma = 0 \tag{6.10a}$$

$$Q'' + (n-1)z^{-1}Q' + [\mu - \tfrac{1}{4}z^2]Q + cQ(PQ)^\sigma = 0 \tag{6.10b}$$

and the scaling (or dilational) symmetry reduction

$$u(\rho,t) = t^{-1/(2\sigma)}P(z)\exp\{\mathrm{i}(\tfrac{1}{8}z^2 - \mu\ln t)\} \tag{6.11a}$$

$$v(\rho,t) = t^{-1/(2\sigma)}Q(z)\exp\{-\mathrm{i}(\tfrac{1}{8}z^2 - \mu\ln t)\} \tag{6.11b}$$

where $z = \rho/t^{1/2}$ and $P(z)$ and $Q(z)$ satisfy

$$P'' + (n-1)z^{-1}P' - [\mu - \tfrac{1}{16}z^2]P + cP(PQ)^\sigma = 0 \tag{6.12a}$$

$$Q'' + (n-1)z^{-1}Q' - [\mu - \tfrac{1}{16}z^2]Q + cQ(PQ)^\sigma = 0. \tag{6.12b}$$

It is easily shown that (6.10) and (6.12) are not of Painlevé type.

Hence we have demonstrated that the $(n+1)$-dimensional nonlinear Schrödinger equation (6.2) can have symmetry reductions associated with a conformal point symmetry only in the case of the critical dimension (6.6).

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